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A Short Proof of the Existence of Highly Chromatic Hypergraphs without Short Cycles

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Using the operation of amalgamation we prove that for every k, n, p there exists a k-graph G (i.e., a k-uniform hypergraph) without cycles of length $\leqslant p$ such that the chromatic number of G is at least n.

INTRODUCTION

The existence of highly chromatic graphs and hypergraphs is a combinatorial classic: It started with Tutte and Zykov in the 1940's [1, 9] and it ended in the 1960's with probabilistic proof of Erdös [2] and Erdös and Hajnal [3] and with a construction of Lovász [4]. The purpose of this note is to give a simple proof of the latter construction, which we state as

THEOREM. *For all positive integers k, n, p there exists a k-graph $(X, \mathscr{M})$ such that*

(1) *$(X, \mathscr{M})$ does not contain a cycle of length $\leqslant p$;*

(2) *$\chi(X, \mathscr{M}) \geqslant n$.*

Here a cycle of length p in $(X, \mathscr{M})$ is a sequence $M_1 ,..., M_p$ of different edges of $\mathscr{M}$ such that there are different vertices

$$x_i \in M_i \cap M_{i+1} \qquad \text{for} \quad i \in \{1, 2,..., p-1\}, x_p \in M_1 \cap M_p .$$

Our proof uses systematically the operation of amalgamation of hypergraphs (defined below) and it may be viewed as an extension of Tutte's idea [1]. Let us remark that (again, as in Lovasz [4]) to prove the existence of highly chromatic graphs without short cycles we have to use the same theorem for hypergraphs. There exists no "purely" graph-theoretical proof of this fact (see [5, 8]).

The present proof arose in the study of Ramsey properties of hypergraphs

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([6], where the notion of an a-partite hypergraph is introduced) and, implicitly, it is contained in our paper on lattices [7].

PRELIMINARIES

Let X_i, $i \in [1, a] = \{1, 2,..., a\}$ be disjoint nonvoid sets, $\bigcup_{i=1}^{a} X_i = X$. An *a-partite k-graph* is a couple $((X_i; i \in [1, a]), \mathscr{M})$, where

(1) $(X, \mathscr{M})$ is a k-graph (i.e., $M \in \mathscr{M} \Rightarrow |M| = k$);

(2) $|M \cap X_i| \leqslant 1$ for every $M \in \mathscr{M}$ and $i \in [1, a]$.

X_i is called the ith parameter. We shall abbreviate $(X_i; i \in [1, a])$ as (X_i). Let $((X_i), \mathscr{M})$ be an a-partite k-graph, $|X_r| = K$, for a fixed $r \in [1, a]$. Let $(Y, \mathscr{N})$ be a K-graph. Denote by $(Y, \mathscr{N}) * ((X_i), \mathscr{M}) = ((X_i'), \mathscr{M}')$ the a-partite k-graph defined by: $X_r' = Y$, $X_i' = X_i \times \mathscr{N}$ for $i \neq r$; $M' \in \mathscr{M}'$ if and only if there exist $N \in \mathscr{N}$ and $M \in \mathscr{M}$ such that $M' \cap X_r' = \iota_N(M \cap X_r)$, $M' \cap X_i' = (M \cap X_i, N)$ for $i \neq r$, where $\iota_N : X_r \rightarrow N$ is for every $N \in \mathscr{N}$ a fixed bijection. The edge M' will be denoted by (M, N).

Consequently $(Y, \mathscr{N}) * ((X_i), \mathscr{M})$ arises by considering $|\mathscr{N}|$ identical copies of $((X_i), \mathscr{M})$ and then identifying the copies of X_r with the edges of $(Y, \mathscr{N})$. This operation is clearly an amalgamation of a very special type.

Proof of the theorem. For fixed n, k we prove the theorem by induction on p. For $p = 1$ the statement of Theorem is true as we have no restriction on cycles. Let $p > 1$ and let the statement of the theorem be true for every $1 \leqslant p' < p$. Put $a = (k - 1) n + 1$ and let $((X_i^1), \mathscr{M}^1)$ be such an a-partite k-graph that for every $\omega \subseteq [1, a]$, $|\omega| = k$, there exists an edge $M \in \mathscr{M}^1$, $M \cap X_i^1 \neq \varnothing$ for all $i \in \omega$. Obviously $((X_i^1), \mathscr{M}^1)$ may be taken without cycles of length $\leqslant p$ (in fact it may be taken without cycles at all).

Define inductively a-partite k-graphs $((X_i^t), \mathscr{M}^t)$, $t \in [2, a + 1]$ as follows: Let $((X_i^{t-1}), \mathscr{M}^{t-1})$ be given. Put $|X_{t-1}^{t-1}| = K_{t-1}$ and let $(Y^{t-1}, \mathscr{N}^{t-1})$ be a K_{t-1} graph without cycles of length $\leqslant p - 1$ and with the chromatic number $\geqslant n$. (Such a hypergraph exists by the induction hypothesis.) Put $((X_i^t), \mathscr{M}^t) = (Y^{t-1}, \mathscr{N}^{t-1}) * ((X_i^{t-1}), \mathscr{M}^{t-1})$ and $((X_i), \mathscr{M}) = ((X^{a+1}), \mathscr{M}^{a+1})$. $((X_i), \mathscr{M})$ (more precisely $(\bigcup_{i=1}^{a} X_i, \mathscr{M})$) is the desired k-graph.

CLAIM 1. *$((X_i^j), \mathscr{M}^j)$ does not contain cycles of length $\leqslant p$ for every $j \in [1, a + 1]$.*

Proof. is simple by induction on j: If $((X_i^j), \mathscr{M}^j)$ does not contain a cycle of length $\leqslant p$ and $((X_i^{j+1}), \mathscr{M}^{j+1})$ does contain such a cycle $C = \{M_1,..., M_q\}$, $q \leqslant p$, then the edges $M_1,..., M_q$ are of the form $(M_1', N_1),..., (M_q', N_q)$ and (1) not all $N_1,..., N_q$ are equal, (2) not all $N_1,..., N_q$ are different—that follows from the definition of $((X_i^{j+1},), \mathscr{M}^{j+1})$. As C is a cycle, $N_1, \ldots, N_q$ contains also a cycle in $(Y^i, \mathscr{N}^j)$ of length $\leqslant q-1$. This is a contradiction.

CLAIM 2. $\chi((X_i), \mathscr{M}) > n$.

Proof. Let $c\colon \bigcup_{i=1}^{a} X_i = X \to [1, n]$ be a fixed coloring. By the definition of $((X_i), \mathscr{M}) = ((X_i^{a+1}), \mathscr{M}^{a+1}) = (Y^a, \mathscr{N}^a) * ((X_i{}^a), \mathscr{M}^a)$ there exists an $N^a \in \mathscr{N}^a$ such that $|\, c(N^a)| = 1$. Consider the coloring (induced by c) of the set $(X_{a-1}^a; N^a) = \{(x, N^a); x \in X_{a-1}^a\}$. As $((X_i{}^a), \mathscr{M}^a) = (Y_i^{a-1}, \mathscr{N}^{a-1}) * ((X_i^{a-1}), \mathscr{M}^{a-1})$ there exists a set $N^{a-1} \subseteq X_{a-1}^a$, $N^{a-1} \in \mathscr{N}^{a-1}$, such that $|\, c(N^{a-1}; N^a)| = 1$. Next consider the coloring of the set $(X^{a-1}; N^{a-1}, N^a) = \{(x, N^{a-1}, N^a); x \in X_{a-2}^{a-1}\}$. Again we find a set $N^{a-2} \in \mathscr{N}^{a-2}$ such that $|\, c(N^{a-2}; N^{a-1}, N^a)| = 1$. By this procedure we find, subsequently, for every $i \in [1, a]$, an edge $N^i \in \mathscr{N}^i$ such that $|\, c(N^i; N^{i+1}, \ldots, N^a)| = 1$. Put $Z_i = (N^i; N^{i+1}, \ldots, N^a)$, and observe $Z_i \subseteq X_i$. Moreover, by the induction definition of $((X_i), \mathscr{M})$, the subhypergraph of $((X_i), \mathscr{M})$ generated by the set $\bigcup_{i=1}^{a} Z_i$ contains an isomorphic copy of $((X_i{}^1), \mathscr{M}^1)$. This leads to a contradiction as (by choice of a) there exists $\omega \subseteq [1, a]$ with $|\, c(\bigcup_{i \in \omega} Z_i)| = 1$ and (by choice of $\mathscr{M}^1$) there exists $M \in \mathscr{M}$ with $M \subseteq \bigcup_{i \in \omega} Z_i$.

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